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COMMENT

Second-order differential equations and non-conservative Lagrangian mechanics

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Abstract. We give an extension to non-conservative Lagrangian systems of a previous work published in this journal.

In a recent paper published in this journal, Sarlet *et al* (1984) proposed a new look at the geometrical formulation of Lagrangian theory. Their basic idea consists of establishing a relation in which a second-order differential equation is a Lagrange equation. The authors consider a special set of 1-forms, denoted here by Δ^1_ξ and defined by

$$\Delta^1_\xi = \{\alpha \in \Delta^1(TM);\ (\mathscr{L}_\xi \circ J^*)(\alpha) = \alpha\}$$

where ξ is a second-order differential equation, $\Delta^1(TM)$ is the set of all 1-forms on TM, the tangent bundle of a m-dimensional manifold M, $\mathscr{L}_\xi$ is the Lie derivative and J^* is the adjoint endomorphism on $\Delta^1(TM)$ induced by the almost tangent structure $J: T(TM) \to T(TM)$ on the double tangent bundle $T(TM)$ of M.

If we associate to each ξ an appropriate 1-form α belonging to Δ^1_ξ, then we may obtain the Lagrange equations of motion. For this α must be exact, $\alpha = \mathrm{d}L$, and L must be a regular function on TM.

It is the purpose of the present comment to enlarge such a point of view to the non-conservative regular situation.

Let us first recall that a mechanical system $\mathscr{M}$ is a triple (M, F, Γ), where M is a smooth finite manifold of dimension m, F is a smooth function on TM and Γ is a semibasic Pfaff form on TM, called the force field. Suppose that the closed 2-form $\omega_F = -\mathrm{d}\mathrm{d}_J F$ on TM is symplectic. Then it can be shown that there is a unique semispray ξ satisfying the equation

$$i_\xi \omega_F = \mathrm{d}E_F + \Gamma \qquad (1)$$

where $E_F = V(F) - F$ is the energy of F. In local coordinates (q^A, v^A) expression (1) assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial v^A}\right) - \frac{\partial F}{\partial q^A} = -\chi_A. \qquad (2)$$

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A mechanical system $\mathscr{M}$ is conservative if the force field Γ is a closed semibasic form. If there is a smooth function $U: TM \to R$ such that $\Gamma = p_M^*(\mathrm{d}U)$ (where $p_M: TM \to M$ is the canonical projection) then $\mathscr{M}$ is said to be a Lagrangian system. In such a case (2) assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v^A}\right) - \frac{\partial L}{\partial q^A} = 0$$

where L is defined by $L = F + U \circ p_M$.

In this comment we will consider non-conservative mechanical systems, i.e. where the force field Γ is not closed. As the results obtained are similar to those of Sarlet *et al* we will discuss here only the main modifications.

Let ξ be a semispray on *TM*, i.e. a second-order differential equation. We associate to ξ a R-bilinear operator

$$\Delta^1(TM) \times \Delta^1(TM) \xrightarrow{E_{\xi,J}} \Delta^1(TM)$$

defined by $E_{\xi,J}(\alpha, \beta) = E_\xi \alpha + J^*\beta$. Now, let us put

$$\Delta^1_{\xi,J} = \ker E_{\xi,J}$$

i.e.

$$\Delta^1_{\xi,J} = \{(\alpha, \beta) \in \Delta^1(TM) \times \Delta^1(TM);\ E_\xi \alpha = -J^*\beta\}.$$

The elements of $\Delta^1_{\xi,J}$ are locally of the form

$$\begin{aligned} \alpha &= (\xi(\bar{\alpha}_A) + \bar{\beta}_A)\, \mathrm{d}q^A + \bar{\alpha}_A\, \mathrm{d}v^A \\ \beta &= \beta_A\, \mathrm{d}q^A + \bar{\beta}_A\, \mathrm{d}v^A. \end{aligned} \tag{3}$$

We note that $E_{\xi,J}(f\alpha, f\beta) = (\xi f)(J^*\alpha) + fE_{\xi,J}(\alpha, \beta)$. Hence $\Delta^1_{\xi,J}$ is a real vector space but is not a module over the ring of functions. It is, however, a module over the ring of the constants of motion. Next we shall relate Δ^1_ξ and $\Delta^1_{\xi,J}$. Let $j: \Delta^1(TM) \to \Delta^1(TM) \times \Delta^1(TM)$, $j(\alpha) = (\alpha, 0)$, be the canonical injection. Then $E_{\xi,J} \circ j = E_\xi$ and so $j(\Delta^1_\xi) \subset \Delta^1_{\xi,J}$.

Definition. A pair of 1-forms (α, β) is said to be *regular* if $(\alpha, \beta) \in \Delta^1_{\xi,J}$ and the 2-form $\omega_\alpha = -\mathrm{d}(J^*\alpha)$ is symplectic. The form β is called the *force field.* A semispray ξ is called a *non-conservative* vector field if there is a regular pair $(\alpha, \beta) \in \Delta^1_{\xi,J}$ such that α is exact, say $\alpha = \mathrm{d}F$.

The condition $E_\xi(\mathrm{d}F) = -J^*\beta$ which characterises ξ may be rewritten in the form

$$i_\xi \omega_F = \mathrm{d}E_F + \Gamma \qquad (\omega_F \overset{\text{def}}{=} \omega_{\mathrm{d}F}) \tag{4}$$

where $\Gamma = -J^*\beta$ is a semibasic form on *TM* and, taking into account (3), from (4) we deduce

$$\xi\left(\frac{\partial F}{\partial v^A}\right) - \frac{\partial F}{\partial q^A} = -\bar{\beta}_A \qquad 1 \leqslant A \leqslant m$$

which yields a system of Lagrange equations with non-conservative forces. Similar results presented by Sarlet *et al* may be re-obtained. For instance, it is easy to see that 'if R is a tensor field of type (1.1) on *TM* satisfying $JR = RJ$ and $J(\mathscr{L}_\xi R) = 0$, then $R^* \circ E_{\xi,J} = E_{\xi,J} \circ (R^* \times R^*)$ and so $R^* \times R^*$ preserves $\Delta^1_{\xi,J}$'.

We may also introduce a kind of dual of $\Delta^1_{\xi,J}$ (see Sarlet *et al* 1984). We define a subset of $\chi(TM)$ by

$$\chi_\xi = \{Y \in \chi(TM);\ J[\xi, Y] = 0\}.$$

Then χ_ξ is a real vector space, but as $J[\xi, fY] = [\xi f, JY] + fJ[\xi, Y]$ one has that χ_ξ is not a $\mathscr{C}^\infty(TM)$ module. However, it is a module over the constants of motion. Locally the elements of χ_ξ have the form

$$Y = Y^A \frac{\partial}{\partial q^A} + \xi(Y^A)\frac{\partial}{\partial v^A}.$$

If we define $\langle Y, (\alpha, \beta)\rangle = (\langle Y, \alpha\rangle, \langle Y, \beta\rangle)$ then we have

$$\langle Y, (\alpha, \beta)\rangle = ((\xi(\bar{\alpha}_A) + \bar{\beta}_A)Y^A + \bar{\alpha}_A\xi(Y^A), \beta_A Y^A + \bar{\beta}_A\xi(Y^A)) \tag{5}$$

for all $(\alpha, \beta) \in \Delta^1_{\xi,J}$, $Y \in \chi_\xi$. As in the conservative situation, we deduce that if (5) holds for all $(\alpha, \beta) \in \Delta_{\xi,J}$ then Y must be in χ_ξ and $(\alpha, \beta) \in \Delta^1_{\xi,J}$.

To end this comment let us examine symmetries which are not of point type, i.e. vector fields Y on TM satisfying $[\xi, Y] = 0$. We recall that for any vector field Y on TM there is a unique tensor field R_Y of type (1.1) on TM such that $(R_Y - \mathscr{L}_Y J) \circ J = 0$, $R_Y J = JR_Y$ and $J(\mathscr{L}_\xi R_Y) = 0$. The tensor R_Y is given by

$$R_Y = (\mathscr{L}_Y J)(\mathscr{L}_\xi J) + (\mathscr{L}_{[\xi, J]} J)J.$$

Proposition. If Y is a symmetry of ξ and $(\alpha, \beta) \in \Delta^1_{\xi,J}$ satisfies

$$J^* R^*_Y \alpha = J^* \,\mathrm{d}f \qquad R^*_Y \beta = \mathrm{d}g \tag{6}$$

for some $f, g \in \mathscr{C}^\infty(TM)$, then

$$(\alpha', \beta') \in \Delta^1_{\xi,J}$$

where $\alpha' = \mathscr{L}_Y \alpha - \mathrm{d}(\xi f)$ and $\beta' = \mathscr{L}_Y \beta - \mathrm{d}(\xi g)$.

Proof. From the above definition of $\alpha'\beta'$, we have

$$\begin{aligned} J^*\alpha' &= J^*(\mathscr{L}_Y \alpha) - J^*\,\mathrm{d}(\xi f) \\ &= \mathscr{L}_Y(J^*\alpha) - (\mathscr{L}_Y J)^*\alpha - J^*\,\mathrm{d}(\xi f) \end{aligned}$$

since $(\mathscr{L}_Y J)^* = \mathscr{L}_Y \circ J^* - J^* \circ \mathscr{L}_Y$. Therefore we have

$$\begin{aligned} \mathscr{L}_\xi(J^*\alpha') &= \mathscr{L}_\xi(\mathscr{L}_Y(J^*\alpha)) - \mathscr{L}_\xi((\mathscr{L}_Y J)^*\alpha) - \mathscr{L}_\xi(J^*\,\mathrm{d}(\xi f)) \\ &= \mathscr{L}_Y(\mathscr{L}_\xi(J^*\alpha)) - \mathscr{L}_\xi((\mathscr{L}_Y J)^*\alpha) - \mathscr{L}_\xi(J^*\,\mathrm{d}(\xi f)) \end{aligned}$$

(since $\mathscr{L}_Y \circ \mathscr{L}_\xi = \mathscr{L}_\xi \circ \mathscr{L}_Y$)

$$= \mathscr{L}_Y \alpha + \mathscr{L}_Y(J^*\beta) - \mathscr{L}_\xi((\mathscr{L}_Y J)^*\alpha) - \mathscr{L}_\xi(J^*\,\mathrm{d}(\xi f)). \tag{7}$$

On the other hand, $(R^*_Y\alpha, R^*_Y\beta) \in \Delta^1_{\xi,J}$; then we have

$$R^*_Y\alpha = \mathscr{L}_\xi(J^* R^*_Y \alpha) - J^* R^*_Y \beta = \mathscr{L}_\xi(J^*\,\mathrm{d}f) - J^*(\mathrm{d}g).$$

The vector field Y is a symmetry of ξ, therefore $Y \in \chi_\xi$. In such a case $R_Y = (\mathscr{L}_Y J)(\mathscr{L}_\xi J)$, because

$$(\mathscr{L}_{[\xi, Y]} J)J = -J(\mathscr{L}_{[\xi, Y]} J) = 0$$

since $J(\mathscr{L}_Z J)$ vanishes for all $Z \in \chi(TM)$. Therefore we deduce that

$$R_Y^*\alpha = (\mathscr{L}_\xi J^*)(\mathscr{L}_Y J^*)\alpha = (\mathscr{L}_\xi J)^* \,\mathrm{d}f + J^* \mathscr{L}_\xi(\mathrm{d}f) - J^*(\mathrm{d}g).$$

Operating on both sides with $(\mathscr{L}_\xi J)^*$, we see that

$$(\mathscr{L}_Y J)^* \alpha = \mathrm{d}f + J^* \,\mathrm{d}(\xi f) + J^*(\mathrm{d}g). \qquad (8)$$

Therefore, taking the Lie derivative of (8) with respect to ξ, we obtain

$$\mathscr{L}_\xi(J^*\alpha') = \mathscr{L}_Y \alpha + \mathscr{L}_Y(J^*\beta) - \mathrm{d}(\xi f) - \mathscr{L}_\xi(J^* \,\mathrm{d}g). \qquad (9)$$

With (9), the relation (7) reduces to

$$\mathscr{L}_\xi(J^*\alpha') = \mathscr{L}_Y \alpha + \mathscr{L}_Y(J^*\beta) - \mathrm{d}(\xi f) - \mathscr{L}_\xi(J^* \,\mathrm{d}g).$$

Thus we have

$$\begin{aligned}\alpha' - \mathscr{L}_\xi(J^*\alpha') &= -\mathscr{L}_Y(J^*\beta) + \mathscr{L}_\xi(J^* \,\mathrm{d}g)\\ &= -(\mathscr{L}_Y J)\beta^* - J^*(\mathscr{L}_Y \beta) + (\mathscr{L}_\xi J)^* \,\mathrm{d}g + J^* \,\mathrm{d}(\xi g).\end{aligned}$$

But from (6) we deduce

$$(\mathscr{L}_\xi J)^*(\mathscr{L}_Y J)^*\beta = \mathrm{d}g.$$

Now, operating again on both sides with $(\mathscr{L}_\xi J)^*$, we obtain

$$(\mathscr{L}_Y J)^*\beta = (\mathscr{L}_\xi J)^* \,\mathrm{d}g.$$

Hence

$$\alpha' - \mathscr{L}_\xi(J^*\alpha') = -J^*(\mathscr{L}_Y \beta - \mathrm{d}(\xi g))$$

or, equivalently,

$$E_\xi \alpha' = -J^*\beta'$$

and so $(\alpha', \beta') \in \Delta^1_{\xi, J}$.

We invite the reader to establish a generalisation of Noether's theorem for the present situation, as was proposed by Sarlet *et al.*

Reference

Sarlet W, Cantrijn F and Crampin M 1984 *J. Phys. A: Math. Gen.* **17** 1999–2009